Math 121 1.4 Functions: Polynomial, Rational, Exponential

Objectives

1) Identify the following types of functions by looking at the algebra:
   - Polynomial $p(x)\ a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$
   - Rational $\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials
   - Exponential $f(x) = b^x$ where b is a constant
   - Piecewise Linear $f(x) = \begin{cases} \text{expression1} & x < a \text{ or } x \leq a \\ \text{expression2} & x > a \text{ or } x \geq a \end{cases}$
   - Absolute Value $f(x) = |\text{expression}|$

2) Graph piecewise linear functions + absolute value functions on paper.

3) Use graphs to find domain and range

4) Evaluate all types of functions.

5) Notice that some rational functions simplify and have holes.

6) Solve equations by factoring, including factoring fractional exponents.

7) Verify solutions of equations graphically, using the x-intercept method.

8) Recognize shifts (also called translations) of a base graph $f(x)$
   - $f(x) + a$ } vertical
   - $f(x) - a$
   - $f(x + a)$ } horizontal
   - $f(x - a)$

9) Find the composition of two functions algebraically.

10) Find and simplify the difference quotient $\frac{f(x+h) - f(x)}{h}$

HONORS 11) Exponential Regression

Polynomial Functions $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$

where all a_i are real numbers

n is a natural number – positive, no fractions or decima

(1) ex. $f(x) = -3x^9 - 5x^8 + 3x^4 - 2$ is a polynomial of degree 8.

(2) ex. $f(x) = 4x^2 - \frac{2}{5}x - 17$ is a polynomial of degree 2 (also called a quadratic)

(3) ex. $f(x) = -\frac{1}{3}x + 7$ is a polynomial of degree 1 (also called a linear)

(4) ex. $f(x) = 6$ is a polynomial of degree 0 (also called a constant linear function)

The domain of any polynomial function is all real numbers.

Rational Functions: $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions

(5) ex. $f(x) = \dfrac{5x-7}{x+3}$

(7) $h(x) = \dfrac{1}{x^2+1}$

(6) ex. $g(x) = \dfrac{x^2-4}{x-2}$

The domain of a rational function is all real numbers except those which make the denominator equal to 0.

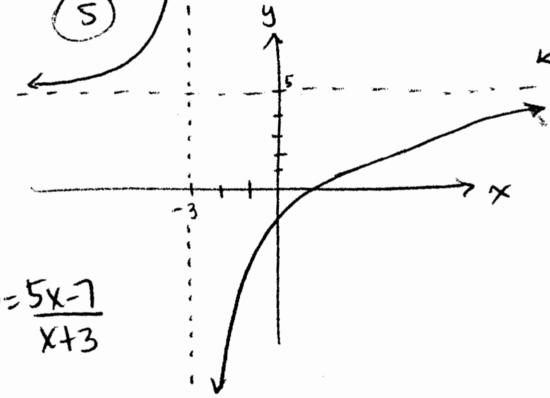
The domain of (5). $\{x : x \in \mathbb{R}, x \neq -3\}$

domain of (6) $\{x : x \in \mathbb{R}, x \neq 2\}$

domain of (7) $\{x : x \in \mathbb{R}\}$ $x^2+1 \neq 0$ because $x^2 = -1$ imaginary

Consider the graphs

(5)



horizontal asymptote $y=5$

$f(x) = \dfrac{5x-7}{x+3}$

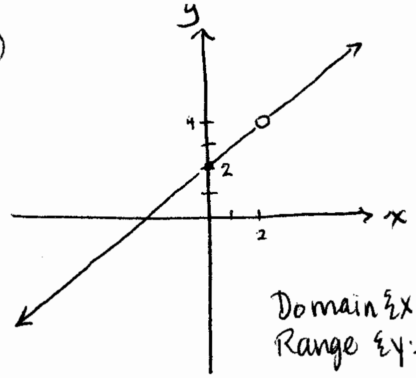
The graph approaches these imaginary lines, but does not touch them.

vertical asymptote $x=-3$

Range $\{y : y \in \mathbb{R}, y \neq 5\}$

(6)

$g(x) = \frac{x^2-4}{x-2}$



$g(x) = \frac{(x^2-4)}{(x-2)} = \frac{(x+2)(x-2)}{(x-2)} = x+2$ for $x \neq 2$

when a factor cancels out

$\frac{x-2}{x-2} = 1$

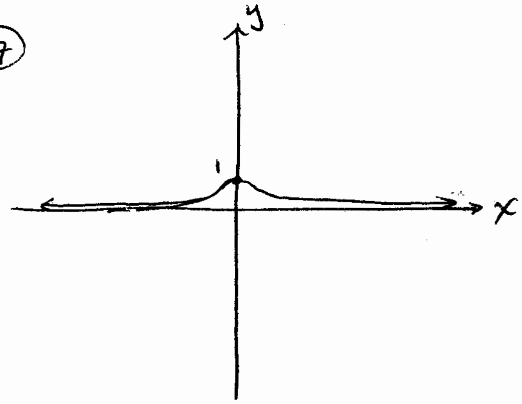
there is a hole in the graph.

Domain $\{x : x \in \mathbb{R}, x \neq 2\}$

Range $\{y : y \in \mathbb{R}, y \neq 4\}$

(7)

$h(x) = \frac{1}{x^2+1}$



Domain $\{x : x \in \mathbb{R}\}$

Range $\{y : 0 < y \leq 1\}$

Exponential Functions are functions of the form

$f(x) = a \cdot b^x + c$ where a, b and c are constants and the variable x is the exponent.

(8) ex. $f(x) = 2^x$

(9) ex. $g(t) = 4000(1.03)^{20t}$ · The compound interest formula! $A = P(1+\frac{r}{n})^{nt}$

(9) ex. $h(x) = \pi^x$ π is an irrational number $\pi \approx 3.14159265358979323846264 33\ldots$

(10) ex. $k(x) = e^x$ e is the natural base, always lower case, also an irrational number $e \approx 2.718281828459045235360287 4\ldots$

(11) ex. $m(x) = .6^x$

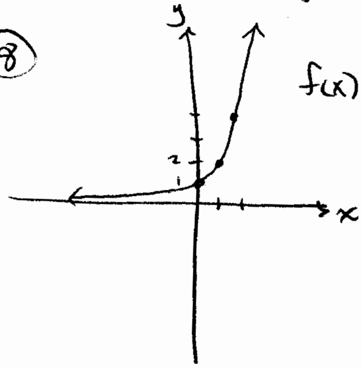
e is the base of the natural logarithm

$\ln(x) = \log_e(x)$

The domain of exponential functions is all real numbers.

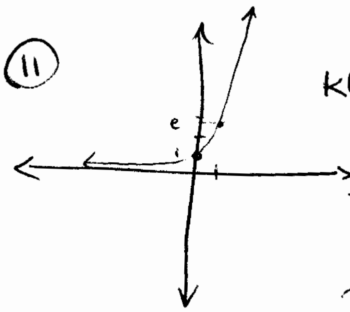
Consider these graphs

⑧ $f(x)=2^x$



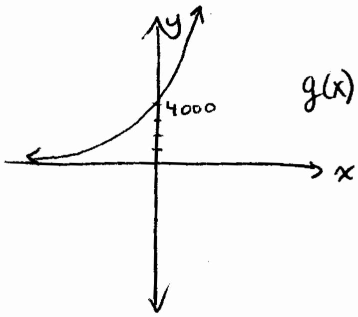
Domain $\{x : x \in \mathbb{R}\}$
Range $\{y : y>0\}$

⑪ $k(x)=e^x$



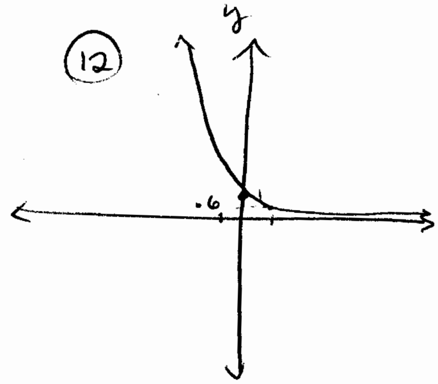
Domain $\{x : x \in \mathbb{R}\}$
Range $\{y : y>0\}$

⑨ $g(x) = 4000\,(1.03)^{20x}$



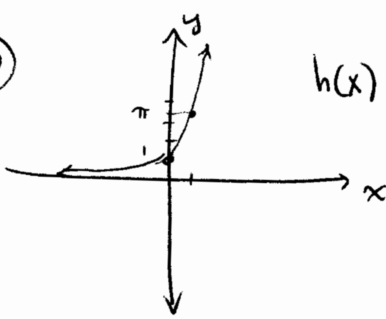
Domain: $\{x : x \in \mathbb{R}\}$
Range: $\{y : y>0\}$

⑫



Domain $\{x : x \in \mathbb{R}\}$
Range: $\{y : y>0\}$

⑩ $h(x) = \pi^x$



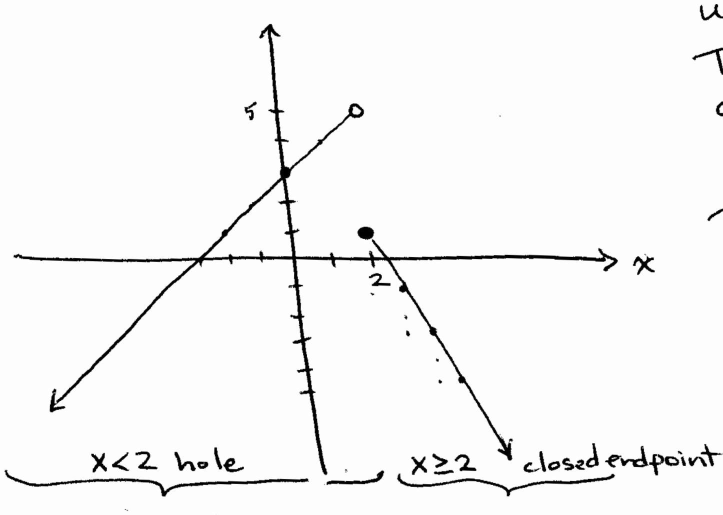
Domain $\{x : x \in \mathbb{R}\}$
Range $\{y : y>0\}$

<u>Piecewise Linear Functions</u> have restrictions on the domain

⑪ ex. $f(x) = \begin{cases} 5-2x & x \geq 2 \\ x+3 & x<2 \end{cases}$

← The expression $5-2x$ is only used for x values greater than or equal to 2.

The expression $x+3$ is only used for x-values less than 2. The find the y-coordinate of the hole, substitute $x=2$.



To graph $5-2x$, subst $x=2$
$5-2(2)=1$, plot $(2,1)$
then use slope -2

Domain: $\{x : x \in \mathbb{R}\}$
Range: $\{y : y<5\}$

(12) ex $g(x) = \begin{cases} 2 & x < 1 \\ \frac{1}{2}x - \frac{3}{2} & x \geq 1 \end{cases}$

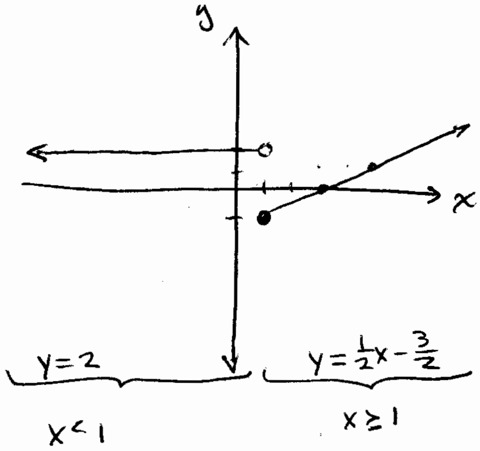
To graph in GC:

$Y_1 = 2(X<1) + (.5X - 1.5)(X \geq 1)$

TEST

$\boxed{\text{2ND}}\boxed{\text{MATH}}$

You still have to do endpoints by hand!



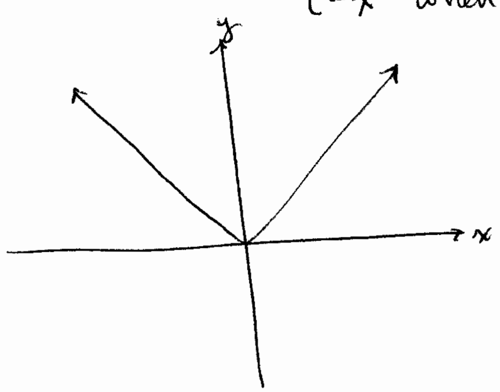
Domain $\{x : x \in \mathbb{R}\}$

Range $\{y : y \geq -1\}$

Absolute Value Functions : Vertical bars

(13) ex. $f(x) = |x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x \leq 0 \end{cases}$

← The opposite of a negative x is positive. ex $-(-5) = 5$



Domain $\{x : x \in \mathbb{R}\}$

Range $\{y : y \geq 0\}$

(14) ex. $g(x) = |2x - 4|$

$2x - 4 = 0$

$2x = 4$

$x = \frac{4}{2} = 2$

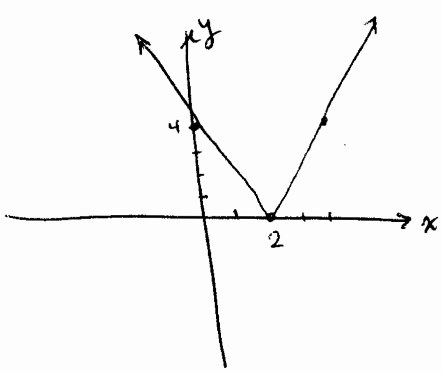
To graph in GC:

$Y_1 = \text{abs}(2X - 4)$

$\boxed{\text{MATH}}$ ▷

MATH $\boxed{\text{NUM}}$ CPX PRB

$\boxed{1}$ abs(



Domain $\{x : x \in \mathbb{R}\}$

Range $\{y : y \geq 0\}$

Solve by factoring. Confirm using x-intercept method on GC.

(15) $2x^5 - 50x^3 = 0$ | (16) $3x^4 + 12x^2 = 12x^3$ | (17) $3x^{7/2} - 12x^{5/2} = 36x^{3/2}$

step1: set equation $= 0$

$3x^4 - 12x^3 + 12x^2 = 0$ | $3x^{7/2} - 12x^{5/2} - 36x^{3/2} = 0$

step2: divide both sides by any GCF coefficient.

$x^5 - 25x^3 = 0$ | $x^4 - 4x^3 + 4x^2 = 0$ | $x^{7/2} - 4x^{5/2} - 12x^{3/2} = 0$

step3: factor out GCF (Greatest Common Factor of variables

$x^3(x^2 - 25) = 0$ | $x^2(x^2 - 4x + 4) = 0$ | $x^{3/2}(x^2 - 4x - 12) = 0$

step4: factor completely

$x^3\underbrace{(x-5)(x+5)}_{\text{difference of squares}} = 0$ | $x^2\underbrace{(x-2)(x-2)}_{\text{perfect square trinomial}} = 0$ | $x^{3/2}\underbrace{(x-6)(x+2)}_{\text{trinomial w/ leading coefficient 1.}} = 0$

-12
-6 2
-4

step5: set each factor equal to zero

$x^3 = 0 \quad x - 5 = 0 \quad x + 5 = 0$

$\boxed{x = 0 \quad x = 5 \quad x = -5}$

$x^2 = 0 \quad x - 2 = 0$

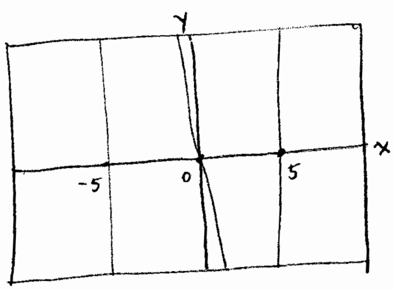
$\boxed{x = 0 \quad x = 2}$

$x^{3/2} = 0 \quad x - 6 = 0 \quad x + 2 = 0$

$\boxed{x = 0 \quad x = 6}$ $x \neq -2$ ↑ extraneous!

Graph in GC

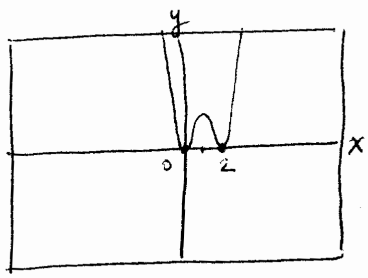
$Y_1 = 2x^5 - 50x^3$



x-intercepts are solutions

$(y = 0)$

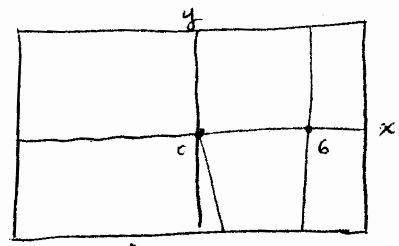
$Y_1 = 3x^4 - 12x^3 + 12x^2$



Y1=3X^(7/2) - 12X^(5/2) - 36X^(3/2)

or

Y1= 3X^3.5 - 12X^(2.5) - 36X^(1.5)



↑ no $x = -2$!?!

$x^{1/2} = \sqrt{x}$

and if $x = -2$ $\sqrt{-2}$ is not real!

(18) Find the composition when $f(x) = 2x-1$ and $g(x) = -3x^2+4x$

a) $f(g(x))$ b) $g(f(x))$ c) $f(f(x))$ d) $g(g(x))$

a) $f(g(x))$ $g(x)$ inside $f(x)$ $\Rightarrow$ replace each x in $f(x)$ by $(-3x^2+4x)$

$= 2(-3x^2+4x) - 1$

$= \boxed{-6x^2 + 8x - 1}$

b) $g(f(x))$ $f(x)$ inside $g(x)$ $\Rightarrow$ replace each x in $g(x)$ by $(2x-1)$

$= -3(2x-1)^2 + 4(2x-1)$

$= -3(2x-1)(2x-1) + 8x - 4$

$= -3(4x^2-4x+1) + 8x - 4$

$= -12x^2 + 12x - 3 + 8x - 4$

$= \boxed{-12x^2 + 20x - 7}$

★ COMPOSITION WILL BE AT THE CORE OF THE CHAIN RULE ★

c) $f(f(x))$ replace each x in $f(x)$ by $(2x-1)$

$= 2(2x-1) - 1$

$= 4x - 2 - 1$

$= \boxed{4x-3}$

d) $g(g(x))$ replace each x in $g(x)$ by $(-3x^2+4x)$

$= -3(-3x^2+4x)^2 + 4(-3x^2+4x)$

$= -3(-3x^2+4x)(-3x^2+4x) - 12x^2 + 16x$

$= -3(9x^4 - 24x^3 + 16x^2) - 12x^2 + 16x$

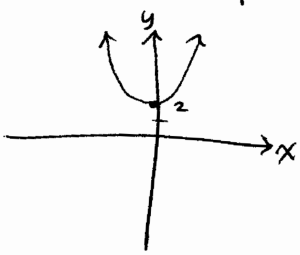
$= -27x^4 + 72x^3 - 48x^2 - 12x^2 + 16x$

$= \boxed{-27x^4 + 72x^3 - 60x^2 + 16x}$

When finding compositions: simplify by FOIL or multiply or distribute, then combine like terms. Don't factor again.

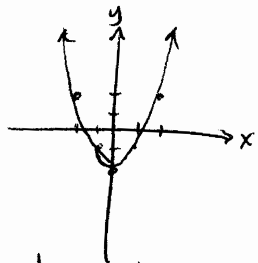
(19) For $f(x) = x^2$ find and graph

a) $f(x)+2$
$= \boxed{x^2+2}$



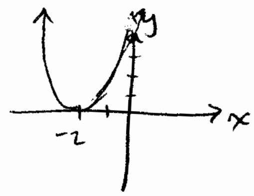
up two units

b) $f(x)-2$
$= \boxed{x^2-2}$



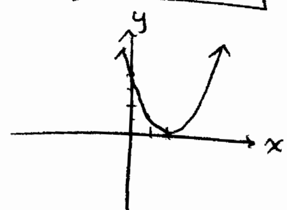
down two units.

c) $f(x+2)$
$= (x+2)^2$
$= \boxed{x^2+4x+4}$



left two units

d) $f(x-2)$
$= (x-2)^2$
$= \boxed{x^2-4x+4}$



right two units.

These are called translations or shifts, and it works the same for any function, and any value $a \geq 0$

$f(x)+a$	up a units	vertical
$f(x)-a$	down a units	
$f(x+a)$	left a units	horizontal
$f(x-a)$	right a units.	

$f(x+h)$ then ...
is a horizontal shift.

(20) Find and simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ for each function

a) $f(x) = 3x^2$

$$\frac{f(x+h)-f(x)}{h} = \frac{3(x+h)^2 - 3x^2}{h}$$ subst $(x+h)$ for x

$$= \frac{3(x^2+2xh+h^2)-3x^2}{h}$$ FOIL $(x+h)^2$

$$= \frac{3x^2+6xh+3h^2-3x^2}{h}$$ dist 3

combine like terms

$$= \frac{6xh}{h} + \frac{3h^2}{h}$$ divide by h.

$$= \boxed{6x+3h}$$

b) $f(x) = 3x^2 - 5x + 2$

$$\frac{f(x+h)-f(x)}{h} = \frac{3(x+h)^2 - 5(x+h) + 2 \quad - \left(3x^2 - 5x + 2\right)}{h}$$

parentheses required! Must distribute negative.

$$= \frac{3(x^2+2xh+h^2) - 5x - 5h + 2 - 3x^2 + 5x - 2}{h}$$

$$= \frac{\cancel{3x^2} + 6xh + 3h^2 \cancel{-5x} - 5h \cancel{+2} \cancel{-3x^2} \cancel{+5x} \cancel{-2}}{h}$$

$$= \frac{6xh + 3h^2 - 5h}{h}$$

$$= \frac{6xh}{h} + \frac{3h^2}{h} - \frac{5h}{h}$$

$$= \boxed{6x + 3h - 5}$$

c) $f(x) = \frac{3}{x}$

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{3}{x+h} - \frac{3}{x}}{h}$$

complex fraction! Two typical methods to simplify

Method 1: Find a common denominator

$$= \frac{\frac{3 \cdot x}{(x+h) \cdot x} - \frac{3 \cdot (x+h)}{x \cdot (x+h)}}{h}$$

$$= \frac{\left(\frac{3x - 3(x+h)}{x(x+h)}\right)}{\left(\frac{h}{1}\right)}$$

← divide fractions

$$= \frac{\cancel{3x} - \cancel{3x} - 3h}{x(x+h)} \div \frac{h}{1}$$

multiply by reciprocal

$$= \frac{-3\cancel{h}}{x(x+h)} \cdot \frac{1}{\cancel{h}}$$

$$= \boxed{\frac{-3}{x(x+h)}}$$

Method 2: Mult by $\frac{LCD}{LCD} = 1$

multiply by 1

$$\frac{x(x+h)}{x(x+h)} = 1$$

$$\frac{\frac{x\cancel{(x+h)}}{1} \cdot \frac{3}{\cancel{x+h}} - \frac{3}{\cancel{x}} \cdot \frac{\cancel{x}(x+h)}{1}}{\frac{h}{1} \qquad \frac{x(x+h)}{1}}$$

$$= \frac{\frac{3x}{1} - \frac{3(x+h)}{1}}{\frac{hx(x+h)}{1}}$$

$$= \frac{3x - 3(x+h)}{hx(x+h)}$$

$$= \frac{3x - 3x - 3h}{hx(x+h)}$$

$$= \frac{-3\cancel{h}}{\cancel{h}x(x+h)}$$

$$= \boxed{\frac{-3}{x(x+h)}}$$